* H E F - 0 0 3 - 1 1 6 1 0 0 1 *

HEF-003-1161001 Seat No. ________

M. Sc. (Sem. I) (CBCS) Examination

November / December - 2017

Mathematics : Paper - CMT - 1001

(Algebra - I) (New Course)

Faculty Code : 003
Subject Code : 1161001

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) Answer **all** the questions.
(2) Each questions carries **14** marks.

1 Attempt any **seven** : **7×2=14**

(1) Let $\phi = (i_1, i_2, i_3) \in S_n$. Prove that $\phi^2 = (i_1, i_3, i_2)$ and $\phi^3 = e$.

(2) Define normal subgroup. Does all the subgroups of $(Z, +)$ are normal subgroups in $(Z, +)$? (Y/N). Justify you answer.

(3) In standard notation write down all the elements of symmetric group S_3 and its subgroup A_3.

(4) Define Sylow p-subgroup of a group G. What is order of Sylow p-subgroup of G ?

(5) Let G be a group, $H < G$ and $a, b, c \in H$. Prove that $\left(ab^{-1}\right)c^{-1} \in H$.

(6) Let R be a ring and I_1, I_2 be two ideals of R. Prove that $I_1 \cap I_2$ is also an ideal of R.

(7) Let R be a ring and I, J be two ideals of R. Define $I + J$ and prove that $I + J$ is also an ideal of R.

(8) Let $\phi : G \to G'$ be a group homomorphism. It is obvious that $Ker\ \phi < G$. Does $Ker\ \phi$, a normal subgroup of G ? Justify your answer.

2 Attempt any **two** : **2×7=14**

(1) Define maximal normal subgroup and simple group. For a group G and $H \Delta G$, prove that H is a maximal normal subgroup of G if and only if G/H is a simple group.

(2) Let H be a group and $K < H$. Let $[H : K] = 2$. Prove that K is a maximal normal subgroup of H.

(3) State and prove first fundamental theorem of group theory.

(4) Let G be an abelian group and $0(G) \geq 2$. Prove that G is simple if and only if G is a cyclic group with prime order.

(5) For $n \geq 5$, prove that the collection of normal subgroups of S_n is $\{\{e\}, A_n, S_n\}$.

3 Attempt any **one** : **1×14=14**

(a) Let $\phi : G \rightarrow G'$ be an onto homomorphism of groups. Prove that :

(i) $H < G \Rightarrow \phi(H) < G'$

(ii) $K < G' \Rightarrow \phi^{-1}(K) < G$

(iii) $H \, \Delta \, G \Rightarrow \phi(H) \, \Delta \, G'$

(iv) $K \, \Delta \, G' \Rightarrow \phi^{-1}(K) \, \Delta G$

(v) $H < G$ and $Ker \, \phi \subseteq H \Rightarrow \phi^{-1}(\phi(H)) = H$

(vi) Let $G \, \{H < G / Ker \, \phi \subseteq H\}$ and D = the collection of all subgroups of G'. Prove that $\psi : C \rightarrow D$ defined by $\psi(H) = \phi(H), \ \forall \ H \in C$ is a bijection.

(b) (i) Let G be a finite group and $P/O(G)$. Prove that $\exists \ g \in G \ni g^p = e$.

(ii) Prove that a group G, with $O(G) = 56$ can not be a simple group.

(c) Let R be a ring with $1 \in R$. Prove that for $n \geq 1$, the ideals of $M_n(R)$ [ring of $n \times n$ matrices over R] are given by $M_n(I)$, where I ranges through all the ideals of R.

4 Attempt any **two** : **2×7=14**

(a) State and prove Sylow's second theorem.

(b) State and prove third isomorphism theorem of ring theory.

(c) Define prime ideal. Let R be a commutative ring, $1 \in R$ and P is an ideal of R with $P \neq R$. Prove that P is a prime ideal of R if and only if R/P is an integral domain.

(d) Let A, B be two ideals of a ring R. Define AB, the product of two ideals. Prove that AB is also an ideal of R.

5 Attempt any **two** : **2×7=14**

(a) Prove that $A_n\ (n \geq 5)$ is a simple group.

(b) State and prove Eisenstein Criterion.

(c) Define action of a group G on a non-empty set X. Prove that a group G acts on itself by the conjugate action

$\phi : G \times G \to G$ defined by

$\phi(g, x) = gxg^{-1}, \ \forall\ g, x \in G.$

(d) (i) Define nil ideal and nilpotent ideal of a ring R.

(ii) Prove that sum of two nilpotent ideals of R is also a nilpotent ideal of R.

(iii) Prove that sum of two nil ideals of R is also a nil ideal of R.